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Asymptotics of Decay of Correlations for Lattice Spin Fields at High Temperatures. I. The Ising Model

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We find the asymptotic decrease of correlations $\langle \sigma_{A+y}, \sigma_B \rangle$, $y \in Z^{\nu+1}$, $|y| \to \infty$, in the Ising model at high temperatures. For the case when monomials σ_A and σ_B both are odd, using the saddle-point method, we find the asymptotics of the correlations for any dimension ν. For even monomials σ_A, σ_B we formulate a general hypothesis about the form of the asymptotics and confirm it in two cases: (1) $\nu = 1$ and the vector y has an arbitrary direction, (2) y is directed along a fixed axis and arbitrary ν. Here we use besides the saddle-point method, some arguments from scattering theory.

KEY WORDS: Ising model; Markov chain; transfer matrix; Friedrichs model; saddle-point method; scattering theory; T-matrix.

1. INTRODUCTION

The decrease of correlations

$$\langle F_{A+y}(\sigma), F_B(\sigma) \rangle = \langle F_{A+y}(\sigma)\, F_B(\sigma) \rangle - \langle F_{A+y}(\sigma) \rangle \langle F_B(\sigma) \rangle$$

$$\text{as} \quad |y| \to \infty \tag{1.1}$$

has been studied in a wide range of papers.[1,2,20–28] Here $\sigma = \{\sigma(x), x \in Z^{\nu+1}\}$ is a Markov Gibbs field on the lattice $Z^{\nu+1}$, $A, B \in Z^{\nu+1}$ are finite subsets of the lattice, F_A, F_B are local functions of the field, $F_A(\sigma) = F_A(\sigma|_A)$ and similarly for the function F_B, $A + y$ is the shift of a set

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A by a vector $y \in Z^{\nu+1}$, and $\langle \cdot \rangle$ is the average with respect to the distribution of the field. Using the arguments of Sinai and Minlos,[1] in a previous paper[2] we proposed a general method to find the asymptotics of the expression (1.1) when $|y| \to \infty$. This method can be applied to the Markov Gibbs field, and it is based on the detailed investigation of the leading branches of the spectrum of the transfer matrix, i.e., the stochastic operator of the corresponding Markov chain. The theory is summarized in refs. 3–5.

In this paper we improve the technical aspects of the method to study cases which were beyond the scope of our previous papers. One of the main improvements is the successful application of the method of scattering theory for the so-called Friedrichs model.[6,7] Here we apply the method for the two-dimensional lattice Ising model and also to the case of the ν-dimensional Ising model with arbitrary ν when y tends to ∞ along some fixed axis of the lattice $Z^{\nu+1}$. This latter case was considered in our previous paper,[2] and in this paper we show that our method can be applied to the case when the vector y tends to ∞ along a direction different from the directions of the coordinate axes, although we have to impose some restrictions on this direction for technical reasons.

To proceed to the description of the method we need some facts concerning the spectral analysis of the transfer matrix for the Ising model. Below we briefly formulate these facts; one can find details and proofs in refs. 3 and 5.

Note that for the two-dimensional Using model ($\nu = 1$) many of our statements can be deduced from the results of Onsager,[8] Kaufmann,[9] Schultz *et al.*,[20] and Evans and Lewis,[27] who found in fact the whole spectrum of the transfer matrix for this model. However, our method is more general because it allows us to study any dimensions for the more complex fields.

Also we remark that most of the papers mentioned above are devoted to the study of the case when the vector y has the direction of some given coordinate axis of the lattice $Z^{\nu+1}$. The case of an arbitrary direction was considered in ref. 28 for $|A| = |B| = 1$ (the two-point correlation function).

Polyakov[23] was the first to establish that the asymptotics of (1.1) as $y \to \infty$ along a given direction has anomalies in the behavior of the preexponential factor in the case when $|A| = |B| = 2$ and for small dimension $\nu = 1, 2$. In this paper we show that the anomalous behavior of the preexponential factor is valid for an arbitrary direction of the vector y in the dimension $\nu = 1$ (see Theorem 2).

Due to space limitations, we postpone some details of the proofs, as well as application of the method to more complex models, a subsequent paper.

2. FORMULATION OF THE PROBLEM AND THE MEAN RESULTS

We consider the $(\nu+1)$-dimensional ferromagnetic Ising model generated by the formal Hamiltonian

$$H(\sigma) = -\sum_{|x-y|=1} \sigma_x \sigma_y, \qquad \sigma = \{\sigma_x = \pm 1, x \in Z^{\nu+1}\}$$

For small values of the parameter β (an inverse temperature) there exists a limiting Gibbs distributions of probabilities μ_β defined on the set $\Omega = \{-1, 1\}^{Z^{\nu+1}}$ of all configurations of the field (for more details, see ref. 10). Denote by $\langle G \rangle = \langle G \rangle_{\mu_\beta}$ the average of a function $G(\sigma)$ defined on Ω with respect to the distribution μ_β. Below we consider functions of the form

$$\sigma_A = \prod_{x \in A} \sigma_x$$

(monomials), where $A \subset Z^{\nu+1}$ is a finite set.

In this paper we investigate the asymptotic behavior of the correlations

$$\langle \sigma_{A+y}, \sigma_B \rangle \qquad \text{as} \quad y \to \infty \tag{2.1}$$

where $y \in Z^{\nu+1}$ is a vector of the lattice $Z^{\nu+1}$, and $A+y$ is a shift of A by the vector y. We define now which sequences $\{y_n\}$, $y_n \to \infty$, we shall consider in this paper. Let $y_0 = (y_0^{(1)},..., y_0^{(\nu+1)}) \in R^{\nu+1}$ be a normalized vector such that

$$|y_0| = \sum_{k=1}^{\nu+1} |y_0^{(k)}| = 1 \tag{2.2}$$

We assume that

$$\frac{y_n^{(k)}}{|y_n|} \to y_0^{(k)}, \quad k = 1,..., \nu+1 \qquad \text{as} \quad n \to \infty \tag{2.3}$$

with the velocity

$$\frac{y_n^{(k)}}{|y_n|} - y_0^{(k)} = o\left(\frac{1}{|y_n|^{1/2}}\right), \qquad n \to \infty \tag{2.4}$$

where $|y_n| = \sum_{k=1,\dots,\nu+1} |y_n^{(k)}|$. In this case we say that the sequence $\{y_n\}$ tends to ∞ along the direction of the vector y_0. For example, the sequence

$$y_n = ([y_0^{(1)} n], \dots, [y_0^{(\nu+1)} n]), \qquad n \to \infty$$

where $[\cdot]$ is the integral part of a number, meets the conditions (2.3), (2.4).

We impose the following conditions on the "directing" vector y_0:

1. The coordinate $y_0^{(\nu+1)}$ is positive and the greatest one

$$y_0^{(\nu+1)} \geqslant |y_0^{(k)}|, \qquad k = 1, \dots, \nu \tag{2.5}$$

This assumption preserves the general of the reasoning. In what follows the direction of the vector $e_{\nu+1} \in Z^{\nu+1}$ will be called the "time" direction.

In addition we require the fulfillment of the following conditions:

2. The following holds:

$$|y_0^{(k)}| < A_\nu, \qquad k = 1, \dots, \nu \tag{2.6}$$

where A_ν a constant defined below, such that $A_\nu < 1/\nu$. The necessity of this condition will be explain below.

3. We have

$$y_0^{(\nu+1)} > \tfrac{1}{3} \qquad \text{(in the context of Theorem 1)} \tag{2.7}$$

$$y_0^{(\nu+1)} > \tfrac{1}{2} \qquad \text{(in the context of Theorem 2)} \tag{2.8}$$

From (2.6)–(2.8) it followers that

$$\frac{|y_0^{(k)}|}{|y_0^{(\nu+1)}|} < 3A_\nu, \qquad k = 1, \dots, \nu$$

or

$$\frac{|y_0^{(k)}|}{|y_0^{(\nu+1)}|} < 2A_\nu, \qquad k = 1, \dots, \nu$$

Hence the vector y_0 lies in a cone enclosing the "time" axis.

Remarks. 1. The conditions (2.7), (2.8) on the coordinate $y_0^{(\nu+1)}$ are necessary for technical reasons. No doubt, the expressions for the asymptotics are true for any vector y_0, but the proof in this case is connected with many technical difficulties. Note that these conditions are important only for large dimensions ν.

2. Note that the symmetry of the model implies that the asymptotics of (2.1) does not depend on the sign of the space coordinates of the vector y_0, so we can suppose that all coordinates $y_0^{(k)}$, $k = 1,\dots, \nu+1$, of the vector are positive.

By virtue of the invariance of the field with respect to the involution

$$\sigma_x \to -\sigma_x, \qquad x \in Z^{\nu+1} \tag{2.9}$$

the correlations (2.1) are equal to zero in the case when the cardinalities $|A|, |B|$ of the sets A and B have different parity. In addition, the asymptotics of (2.1) is distinguished according to the parity of the sets A and B.

We formulate now the main results of this paper.

Theorem 1. Let $|A|$ and $|B|$ both be odd. Then for any vector y_0 satisfying the conditions (2.2) and (2.5)–(2.7) there exists a vector $m = m_\nu(y_0) = \{m^{(k)}(y_0),\ k = 1,\dots, \nu+1\} \in R^{\nu+1}$ such that

$$(m_\nu(y_0), y_0) > 0$$

and for any sequence of the vectors $y = \{y_n\}$ tending to infinity along the direction of the vector y_0 the following asymptotics is true:

$$\langle \sigma_{A+y_n}, \sigma_B \rangle = \frac{C_\nu}{|y_n|^{\nu/2}} e^{-(m_\nu(y_0),\, y_n)}(1 + o(1)) \qquad \text{as} \quad n \to \infty \tag{2.10}$$

Here $C_\nu = C_\nu(A, B, y_0)$ are constants independent of the $\{y_n\}$.

Remark. In the case when

$$\min_{k=1,\dots,\nu} |y_0^{(k)}| \geqslant \alpha > 0$$

where α is an absolute constant, the coordinates of the vector $m_\nu(y_0)$ have the following form

$$m_\nu^{(k)}(y_0) = \operatorname{sign} y_0^{(k)} \cdot \ln \frac{|y_0^{(k)}|}{\beta} + O(\beta), \qquad k = 1,\dots, \nu+1$$

This result is consistent with the formula from ref. 28.

In the case $\nu = 1$ we can write the representation of the vector $m_1(y_0)$ for any y_0:

$$m_1^{(1)}(y_0) = -\ln\{[u^2(1-\beta^2)^2+4]^{1/2} - u(1+\beta^2)\} + \ln 2(1-\beta u) + O(\beta)$$

$$m_1^{(2)}(y_0) = -\ln \beta(1-u\beta) - \ln\{[u^2(1-\beta^2)^2+4]^{1/2} - u((1+\beta^2)\}$$
$$+\ln\{[u^2(1-\beta^2)^2+4]^{1/2}(1+\beta^2) - u(1-2\beta^2) - 4\beta\} + O(\beta)$$

where $u = \beta^{-1}(y_0^{(1)}/y_0^{(2)})$.

In the case when $|A|$ and $|B|$ both are even we suppose that the following general statement holds true, but we can prove this assumption only in some special cases (see below, Theorem 2).

Conjecture. Let $|A|$, $|B|$ both be even. Then under the conditions of Theorem 1 and the condition (2.8) we have the following asymptotic formulas as $n \to \infty$:

$$\langle \sigma_{A+y_n}, \sigma_B \rangle = \frac{B_1}{|y_n|^2} e^{-2(m_1(y_0),\, y_n)}(1+o(1)) \qquad \text{for} \quad \nu = 1 \tag{2.11}$$

$$\langle \sigma_{A+y_n}, \sigma_B \rangle = \frac{B_2}{|y_n|^2 (\ln |y_n|)^2} e^{-2(m_2(y_0),\, y_n)}(1+o(1)) \qquad \text{for} \quad \nu = 2 \tag{2.12}$$

$$\langle \sigma_{A+y_n}, \sigma_B \rangle = \frac{B_\nu}{|y_n|^\nu} e^{-2(m_\nu(y_0),\, y_n)}(1+o(1)) \qquad \text{for} \quad \nu \geqslant 3 \tag{2.13}$$

Here $m_\nu(y_0)$ is the same vector as in Theorem 1, $B_\nu = B_\nu(A, B, y_0)$ and $\nu = 1, 2,\ldots$ are absolute constants.

Theorem 2. The conjecture is true in the following cases:

1. $\nu = 1$.
2. $y_0 = e_{\nu+1}$ for arbitrary ν.

Remarks. 1. The proof of the second statement of Theorem 2 is given in ref. 2. Here we present a briefer proof of this result.

2. The first statement of Theorem 2 can be deduced from the results of refs. 8, 9, and 20, where the spectrum of the transfer matrix in the Ising model for $\nu = 1$ was found. Using this result and Theorem 1, one can obtain after straightforward reasoning the asymptotic formula (2.11). However, in this paper we use another method which can be adapted to the investigation of the general case.

3. PROOF OF THEOREM 1. PRELIMINARY CONSTRUCTIONS AND FACTS

Now we list facts required for the proof of the theorems. Many of them are well known (we shall point out the corresponding references); the proofs of new results are given in Appendices A–E.

Since the random Gibbs field for the Ising model is the Markov one, we can represent it as a Markov chain considering the axis of the vector $e_{\nu+1}$ as the time direction,

$$\sigma_t = \{\sigma_{(\bar{x},t)}, \bar{x} \in Z^\nu\}, \qquad t = ..., -1, 0, 1, 2, ...$$

A space of states of the Markov chain $\Omega_0 = \{-1, 1\}^{Z^\nu}$ is the set of all configurations of the field on the zero-slice

$$Y_0 = \{x = (x^{(1)}, ..., x^{(\nu)}, 0)\}$$

The stochastic semigroup of the operators $T^t (t = 0, 1, ...)$ for the Markov chain is determined in the usual way, and it acts in the Hilbert space H of the functions defined on the set Ω_0 with the scalar product

$$(f_1, f_2)_H = \langle f_1 \cdot \bar{f_2} \rangle$$

The generator of the semigroup is designated by the transfer matrix T of the field. Since the field is invariant with respect to "the inversion of the time," the operator T is self-adjoint, and

$$\langle f_1(\sigma_t) \cdot \bar{f_2}(\sigma_0) \rangle = (T^t f_1, f_2)_H$$

where $f_1, f_2 \in H$, and σ_t is the value of the field at the moment t. We can also introduce in H the unitary group $\{U_x, x \in Z^\nu\}$ of the "spatial shifts" acting in H as follows:

$$(U_x f)(\sigma) = f(\sigma - x)$$

where $f \in H$, $\sigma \in \Omega_0$, and $\sigma - x$ is a shift of the configuration σ by the vector $(-x) \in Y_0$. The operator T commutes with the unitary group $\{U_x\}$.

We shall need the following spectral properties of the operator T (one call find the proofs in refs. 10–13 and 5).

1. For small enough $\beta (0 < \beta < \beta_0)$ the space H is decompose into the direct sum

$$H = H_0 \oplus H_1 \oplus H_2 \oplus H_3 \oplus H_4 \qquad (3.1)$$

of mutually orthogonal subspaces $H_0, ..., H_4$ invariant with respect to the operators T and U_x. Here $H_0 = \{\text{const}\}$ is a space of constants, H_1 is a so-called one-particle subspace, and H_2 is a two-particle subspace. Note that the sum

$$H^{\text{odd}} = H_1 \oplus H_3$$

contains all functions which are odd with respect to the involution (2.9), and the sum

$$H^{\text{even}} = H_0 \oplus H_2 \oplus H_4$$

contains all such even functions. Let

$$T_i = T|_{H_i}, \qquad U_x^{(i)} = U_x|_{H_i}, \qquad i = 1, 2, 3, 4$$

be the restrictions of the operators T and U_x on the subspaces H_i, respectively. Then we have the following estimates for the operators T_3 and T_4:

$$\|T_3\| \leqslant C\beta^3, \qquad \|T_4\| \leqslant C\beta^4 \tag{3.2}$$

where C is an absolute constant. Let us describe first the operators T_1 and $U_x^{(1)}$ in detail.

2. The operators T_1 and $U_x^{(1)}$. There exists a unitary mapping

$$V_1: \quad H_1 \to L_2(T^\nu, d\lambda)$$

of the space H_1 into the Hilbert space $L_2(T^\nu, d\lambda)$ of the functions defined on the ν-dimensional torus T^ν ($d\lambda$ is a normalized Haar measure on T^ν) transforming the operators T_1 and $U_x^{(1)}$ onto operators $\tilde{T}_1$ and $\widetilde{U_x^{(1)}}$, respectively:

$$\begin{aligned} \tilde{T}_1 f(\lambda) &= a(\lambda) f(\lambda), \qquad f \in L_2(T^\nu, d\lambda) \\ \widetilde{U_x^{(1)}} f(\lambda) &= e^{i(x, \lambda)} f(\lambda) \end{aligned} \tag{3.3}$$

where $\lambda = (\lambda^{(1)}, ..., \lambda^{(\nu)})$, $(x, \lambda) = \sum x^{(k)} \lambda^{(k)}$. Here $a(\lambda) = a(\lambda, \beta)$ is a function analytic on T^ν which has an analytic extension to a complex manifold W_β, where W_β is a factor-manifold obtained from a complex region

$$\tilde{W}_\beta = \{\lambda = (\lambda^{(1)}, ..., \lambda^{(\nu)}) \in C^\nu, |\text{Im}\, \lambda^{(k)}| \leqslant G_\nu, k = 1, ..., \nu\} \tag{3.4}$$

as a result of the identification of the points by the group of shifts

$$\lambda \to \lambda + 2\pi k, \qquad k \in Z^\nu, \quad \lambda \in \tilde{W}_\beta$$

Here $G_\nu = |\ln \beta| - \ln D_\nu$, and D_ν is an absolute constant such that $D_\nu > \nu$. Note that the manifold W_β has the natural complex structure inherited from the structure of $\tilde{W}_\beta$. In this case for every point $\lambda \in W_\beta$ we denote a projection of λ on T^ν by $\text{Re}\, \lambda$ and a projection of λ on the "imaginary cube" $(-G_\nu, G_\nu)^\nu$ by $\text{Im}\, \lambda$.

Lemma 3.1. The function $a(\lambda, \beta)$ has the following representation:

$$a(\lambda, \beta) = \beta a_0(\lambda, \beta) + \beta^2 a_1(\lambda, \beta) \tag{3.5}$$

The functions $a_0(\lambda, \beta)$, $a_1(\lambda, \beta)$ are real and even as $\lambda \in T^\nu$, they have an analytic extension in the domain W_β, and the function $a_1(\lambda, \beta)$ is uniformly hounded inside W_β together with all its derivatives:

$$|D^\alpha a_1(\lambda, \beta)| \leqslant C_\alpha$$

where C_α are absolute constants,

$$D^\alpha = \frac{\partial^{|\alpha|}}{(\partial\lambda_1)^{\alpha_1} \cdots (\partial\lambda_\nu)^{\alpha_\nu}}$$

and $\alpha = (\alpha_1, ..., \alpha_\nu)$ is an integer-valued multi-index. The function $a_0(\lambda, \beta)$ in W_β can be written in the form of the series

$$a_0(\lambda, \beta) = \sum_{n = \{n_1, ..., n_\nu\} \in Z^\nu} \beta^{|n|} \frac{|n|!}{\prod |n_k|!} e^{i(n, \lambda)} \tag{3.6}$$

where $|n| = \sum_{k=1,...,\nu} |n_k|$, $\lambda = (\lambda^{(1)}, ..., \lambda^{(\nu)}) \in W_\beta$.

Proof. See Appendix A.

Remark. In the case $\nu = 1$ Schultz *et al.*[20] found the precise representation for the function $a(\lambda, \beta)$:

$$a(\lambda, \beta) = e^{2\beta \cos \lambda} \{(A_\beta - \cos \lambda) - [A_\beta - \cos \lambda)^2 - 1]^{1/2}\} \tag{3.7}$$

where $A_\beta = \text{ch } 2\beta \cdot \text{cth } 2\beta$. Here we take the branch of the square root that has positive values as λ is real. From (3.7) we notice that in the case $\nu = 1$ the function $a(\lambda, \beta)$ is analytic inside the region $|\text{Im } \lambda| < \tau$, where τ is a root of the equation

$$\text{ch } \tau = \frac{1}{\text{sh } 2\beta} + \text{sh } 2\beta - 1$$

From the formulas (3.5), (3 6) and the evenness of the function $a(\lambda, \beta)$ it follows that the maximum of $a(\lambda, \beta)$ on the torus T^ν is attained at the point $\lambda = 0$,

$$\max_{\lambda \in T^\nu} a(\lambda, \beta) = a(0, \beta)$$

Thus the operator $\tilde{T}_1$ has an absolutely continuous spectrum coinciding with the segment

$$[\min_{\lambda \in T^\nu} a(\lambda, \beta), \max_{\lambda \in T^\nu} a(\lambda, \beta) = a(0, \beta)] \tag{3.8}$$

and the norm of $\tilde{T}_1$ has the order β. Note that the length of the segment (3.8) has the order β^2.

Lemma 3.2. 1. The function $a(\lambda, \beta)$ is not equal to 0 inside the domain W_β: $a(\lambda, \beta) \neq 0$, $\lambda \in W_\beta$.

2. The function $a(\lambda, \beta)$ is real as $2 \operatorname{Re} \lambda = 0$, $\lambda \in W_\beta$ (or at the points $\operatorname{Re} \lambda = \pi k$, where $k = (k^{(1)},..., k^{(\nu)})$, $k^{(i)} = 0, 1$, $i = 1,..., \nu$, by the identification of the torus T^ν with the cube $[0, 2\pi]^\nu$), and the second differential of $a(\lambda, \beta)$ is nondegenerate for all $\lambda \in W_\beta$ such that $\operatorname{Re} \lambda = 0$.

3. In the case $\nu = 1$ the second derivative of the function $a(\lambda, \beta)$, $\lambda \in W_\beta$, is not equal to zero at all critical points of the function $a(\lambda, \beta)$.

Proof. See Appendix B.

Lemma 3.3. Let $A \subset Z^\nu$ be a finite set such that $|A|$ is odd. Then the function

$$f_A^{(1)}(\lambda) = (V_1 P_{H_1} \sigma_A)(\lambda) \in L_2(T^\nu, d\lambda) \tag{3.9}$$

has an analytic extension to the region W_β. Here P_{H_1} is a projector from $L_2((-1, 1)^{Z^{\nu+1}}, \mu)$, where μ is the distribution of probabilities for the Ising field on $Z^{\nu+1}$, on the space $H_1 \subset H \subseteq L_2((-1, 1)^{Z^{\nu+1}}, \mu)$.

Proof. The proof can be obtained from the reasoning of refs. 5 and 10, and is based on the constructions of the subspace H_1 and the mapping V_1, as well as on the general estimates for Gibbs fields correlations.[16]

Remark. From the decomposition

$$f(\lambda) = \sum_{n \in Z^\nu} c_n e^{i(n, \lambda)}, \qquad \lambda \in W_\beta$$

which is true for every analytic function in W_β, it follows that the function

$$f^*(\lambda) = \sum_{n \in Z^\nu} \overline{c_n}\, e^{-i(n, \lambda)}$$

is also analytic in W_β. This function is equal to $\overline{f(\lambda)}$ when λ is real, $\lambda \in T^\nu$.

Below we shall describe the operators T_2 and $U_x^{(2)}$ in detail when we prove Theorem 2.

Proof of Theorem 1. Let A and B both be odd. Denote by $A' \subset Z^1$ and $B' \subset Z^1$ the projections of A and B, respectively, on the "time" axis $e_{\nu+1}$, and let $a \in A'$ be the rightmost point of the set A', and $b \in B'$ be the leftmost point of the set B'. We can suppose without loss of generality that $b = 0$. Then

$$\langle \sigma_{A+y}, \sigma_B \rangle = (T^{y^{(\nu+1)}+a} U_{\bar{y}} P_H \sigma_{A-ae_{\nu+1}}, P_H \sigma_B)_H \tag{3.10}$$

when $y^{(\nu+1)} > a$. Here $y = (\bar{y}, y^{(\nu+1)})$, $\bar{y} = (y^{(1)},..., y^{(\nu)}) \in Z^\nu$, and P_H is a projector on the space H. The formula (3.10) follows from the facts that the field is Markov, translation invariant, and invariant with respect to the inversion of the time. From (3.1), (3.2), and Lemma 2.2 it follows that the scalar product in (3.10) can be written as

$$(T_1^{y^{(\nu+1)}+a} U_{\bar{y}}^{(1)} P_{H_1} \sigma_{A-ae_{\nu+1}}, P_{H_1} \sigma_B) + O((C\beta)^{3y^{(\nu+1)}}) \tag{3.11}$$

It turns out that the asymptotics of the first term in (3.11) leads to the formula (2.10); therefore the second term in (3.11) is not essential for the asymptotics provided that the inequality (2.7) is realized. Using the spectral representation (3.3), we have for the first term in (3.11)

$$\begin{aligned}
&(T_1^{y^{(\nu+1)}+a} U_{\bar{y}}^{(1)} P_{H_1} \sigma_{A-ae_{\nu+1}}, P_{H_1} \sigma_B) \\
&\quad = \int_{T^\nu} (a(\lambda))^{y^{(\nu+1)}+a} e^{i(\bar{y},\lambda)} f^{(1)}_{A-ae_{\nu+1}}(\lambda) \overline{f^{(1)}_B}(\lambda)\, d\lambda \\
&\quad = \int_{T^\nu} \exp\{ y^{(\nu+1)} \cdot f_\xi(\lambda)\}\, g(\lambda)\, d\lambda
\end{aligned} \tag{3.12}$$

where the functions $f^{(1)}_{A-ae_{\nu+1}}(\lambda)$ and $f^{(1)}_B(\lambda)$ are defined by the formula (3.9),

$$g(\lambda) = f^{(1)}_{A-ae_{\nu+1}}(\lambda) \cdot \overline{f^{(1)}_B}(\lambda) \cdot (a(\lambda))^a$$

and for any $\xi \doteq (\xi^{(1)},..., \xi^{(\nu)}) \in R^\nu$

$$f_\xi(\lambda) = \ln a(\lambda) + i(\xi, \lambda)$$

It should be assumed that

$$\xi = \xi(y) = \frac{\bar{y}}{y^{(\nu+1)}} \tag{3.13}$$

in formula (3.12). Since $f_A^{(1)}(\lambda)$, $f_B^{(1)}(\lambda)$ are functions analytic in W_β, we can find the asymptotics of the integral (3.12) by the use of the saddle-point method.[17]

Lemma 3.4. For small $\beta (0<\beta<\beta_0)$ and for any $\xi = (\xi^{(1)},..., \xi^{(\nu)}) \in R^\nu$, such that

$$\xi^{(k)} \geqslant 0, \qquad \left(1 + \sum_{s=1,\dots,\nu} |\xi^{(s)}|\right) > D_\nu \cdot |\xi^{(k)}|, \qquad k = 1, 2,..., \nu \quad (3.14)$$

where D_ν is the same constant sac in the definition (3 4), there exists a unique critical point $\lambda_0 = \lambda_0(\xi)$ of the function

$$f_\xi(\lambda) = \ln a(\lambda) + i \sum_{j=1,\dots,\nu} \lambda^{(j)} \xi^{(j)}$$

with pure imaginary coordinates lying in the region W_β. This point is non-degenerate and it is a saddle point for the integral

$$\int_{T^\nu} g(\lambda)\, e^{tf_\xi(\lambda)}\, d\lambda \tag{3.15}$$

The point $\lambda_0(\xi)$ has a differentiable dependence on parameters $\xi = (\xi^{(1)},..., \xi^{(\nu)})$.

Proof. The proof is given in Appendix C.

Remark. Now the condition (2.6) follows from (3.14), (3.13), (2.2), and (2.3), where the constant A_ν should be equal to $A_\nu = 1/D_\nu$.

Recall that the critical point λ_0 is a saddle point of the integral (3.15) if there exists a surface Γ inside W_β (or a contour γ in the case $\nu = 1$) passing through the point λ_0 such that the following conditions hold:

1. We have

$$\int_\Gamma g(\lambda)\, e^{tf_\xi(\lambda)}\, d\lambda = \int_{T^\nu} g(\lambda)\, e^{tf_\xi(\lambda)}\, d\lambda$$

2. We have

$$\max_{\lambda \in \Gamma} \operatorname{Re} f_\xi(\lambda) = \operatorname{Re} f_\xi(\lambda_0)$$

3. In a neighborhood of the point λ_0 the surface Γ passes along the level surface $\operatorname{Im} f_\xi(\lambda) = \text{const}$ [or the surface of steepest, descent of the function $\operatorname{Re} f_\xi(\lambda)$; for more details see ref. 17].

Now we deduce the asymptotics of the integral (3.19) using Lemma 3.4. Let $y_n \to \infty$ be a sequence of vectors tending to infinity along the direction y_0, and let ξ_n be a vector with coordinates

$$\xi_n^{(s)} = \frac{y_n^{(s)}}{y_n^{(\nu+1)}}, \qquad s = 1,..., \nu$$

and $\xi(\infty)$ be a vector with coordinates

$$\xi^{(s)}(\infty) = \frac{y_0^{(s)}}{y_0^{(\nu+1)}}, \qquad s = 1,..., \nu$$

Note that the relation

$$f_{\xi_n}(\lambda_0(\xi_n)) - f_{\xi_n}(\lambda_0(\xi(\infty))) = O(|\xi - \xi(\infty)|^2) = o\left(\frac{1}{|y_n|}\right) \tag{3.16}$$

as $n \to \infty$, follows from Lemma 3.4 and the condition (2.4). Thus, putting $\lambda_0 = \lambda_0(\xi(\infty))$, we can use Lemma 3.4 for the integral (3.12). Going over the surface Γ (or contour γ) constructed in Lemma 3.4, we can apply the Laplace method to the integral (3.12) over Γ, and taking into account (3.16), we obtain the asymptotics (2.10), where

$$m^{(s)}(y_0) = -i\lambda_0^{(s)}(\xi(\infty)), \qquad s = 1,..., \nu$$

$$m^{(\nu+1)}(y_0) = -\ln a(\lambda_0(\xi(\infty)))$$

Theorem 1 is proved.

4. PROOF OF THEOREM 2

Since for even $|A|$ the projection of the monomial $\sigma_A \in L_2((-1, 1)^{Z^{\nu+1}}, \mu)$ on the space H_1 equals zero, and $P_{H_2}\sigma_A \neq 0$, to find the asymptotics (2.1) we have to study the characteristics of the operators T_2 and $U_x^{(2)}$ acting in the space H_2.

Characteristics of the Operators T_2 and $U_x^{(2)}$. There exists a unitary mapping

$$V_2\colon \quad H_2 \to \hat{L}_2^{\text{sym}}(T^\nu \times T^\nu, d\lambda_1\, d\lambda_2) \subset L_2^{\text{sym}}(T^\nu \times T^\nu, d\lambda_1\, d\lambda_2)$$

transforming the operators T_2 and $U_x^{(2)}$ into the operators $\tilde{T}_2$ and $\widetilde{U_x^{(2)}}$:

$$(\tilde{T}_2 f)(\lambda_1, \lambda_2) = a(\lambda_1)\, a(\lambda_2)\, f(\lambda_1, \lambda_2) + \int_{T^\nu \times T^\nu} S(\lambda_1, \lambda_2, \mu_1, \mu_2)\, \delta(\lambda_1 + \lambda_2 - \mu_1 - \mu_2) \times f(\mu_1, \mu_2)\, d\mu_1\, d\mu_2 \tag{4.1}$$

$$(\widetilde{U_x^{(2)}} f)(\lambda_1, \lambda_2) = e^{i(x, \lambda_1 + \lambda_2)} f(\lambda_1, \lambda_2), \qquad x \in Z^\nu \tag{4.2}$$

where $(\lambda_1, \lambda_2) \in T^\nu \times T^\nu$, $f(\lambda_1, \lambda_2) \in \hat{L}_2^{\text{sym}}$, and $\hat{L}_2^{\text{sym}} \subset L_2^{\text{sym}}(T^\nu \times T^\nu, d\lambda_1\, d\lambda_2)$ is the space of symmetric functions $f(\lambda_1, \lambda_2)$ orthogonal to functions of the form $h(\lambda_1 + \lambda_2) \in L_2^{\text{sym}}(T^\nu \times T^\nu, d\lambda_1\, d\lambda_2)$:

$$\int_{T^\nu \times T^\nu} f(\lambda_1, \lambda_2)\, h(\lambda_1 + \lambda_2)\, d\lambda_1\, d\lambda_2 = 0$$

Here the function $a(\lambda)$ is the same one as in (3.3), and the kernel $S(\lambda_1, \lambda_2, \mu_1, \mu_2)$ is an analytic function with respect to each variable $\lambda_1, \lambda_2, \mu_1, \mu_2 \in W_\beta$ which is defined on the manifold

$$\Gamma_\beta = \{(\lambda_1, \lambda_2, \mu_1, \mu_2) \in (W_\beta)^4 \colon \lambda_1 + \lambda_2 - \mu_1 - \mu_2 = 0\} \tag{4.3}$$

In this case the kernel $S(\lambda_1, \lambda_2, \mu_1, \mu_2)$ has the form[15]

$$S(\lambda_1, \lambda_2, \mu_1, \mu_2) = -a(\lambda_1)\, a(\lambda_2) - a(\mu_1)\, a(\mu_2) + \int_{v_1 + v_2 = \lambda_1 + \lambda_2 = \mu_1 + \mu_2} a(v_1)\, a(v_2)\, dv_1\, dv_2 + K(\lambda_1, \lambda_2, \mu_1, \mu_2) \tag{4.4}$$

where the function $K(\lambda_1, \lambda_2, \mu_1, \mu_2)$ analytic in $(W_\beta)^4$ is defined on the manifold (4.3) and satisfies the estimate

$$|K(\lambda_1, \lambda_2, \mu_1, \mu_2)| < C\beta^3, \qquad \lambda_1, \lambda_2, \mu_1, \mu_2 \in T^\nu$$

In addition, the function $K(\lambda_1, \lambda_2, \mu_1, \mu_2)$ meets the following conditions:

$$K(\lambda_1, \lambda_2, \mu_1, \mu_2) = \overline{K(\mu_1, \mu_2, \lambda_1, \lambda_2)}, \qquad \lambda_1, \lambda_2, \mu_1, \mu_2 \in T^\nu$$

$$\iint_{T^\nu \times T^\nu} K(\lambda_1, \lambda_2, \mu_1, \mu_2)\, \delta(\mu_1 + \mu_2 - \Lambda)\, d\mu_1\, d\mu_2 = \iint_{T^\nu \times T^\nu} K(\lambda_1, \lambda_2, \mu_1, \mu_2)\, \delta(\lambda_1 + \lambda_2 - \Lambda)\, d\lambda_1\, d\lambda_2 = 0$$

for every $\Lambda \in T^\nu$.

From (4.1) it follows that we can represent the space H_2 as the direct integral of the Hilbert spaces[14]

$$H_2 = \oint_{T^\nu} H_2(\Lambda)\, d\Lambda \tag{4.5}$$

such that the operators $\tilde{T}_2$ and $\widetilde{U_x^{(2)}}$ have the analogous representation in the form of the direct integrals

$$\tilde{T}_2 = \oint_{T^\nu} \tilde{T}_2(\Lambda)\, d\Lambda, \qquad \widetilde{U_x^{(2)}} = \oint_{T^\nu} \widetilde{U_x^{(2)}}(\Lambda)\, d\Lambda \tag{4.6}$$

In so doing, the operator $\widetilde{U_x^{(2)}}(\Lambda)$ is divisible by the unit operator $E(\Lambda)$ in $H_2(\Lambda)$:

$$\widetilde{U_x^{(2)}}(\Lambda) = e^{i(x,\Lambda)} E(\Lambda) \tag{4.7}$$

and the space $H_2(\Lambda)$ for every $\Lambda \in T^\nu$ is unitary equivalent to the Hilbert space $\hat{L}_2^\Lambda(T^\nu, d\hat{\lambda}) \subset L_2^\Lambda(T^\nu, d\hat{\lambda})$ of the functions $f(\hat{\lambda})$ on the torus T^ν orthogonal to the constants

$$\int_{T^\nu} f(\hat{\lambda})\, d\hat{\lambda} = 0$$

and invariant with respect to the substitution $\hat{\lambda} \to \Lambda - \hat{\lambda}$, $\hat{\lambda} \in T^\nu$. In essence, the decompositions (4.5)–(4.7) and the transfer to the space $\hat{L}_2^\Lambda(T^\nu, d\hat{\lambda})$ signify the passage to the new coordinates on $T^\nu \times T^\nu$

$$\Lambda = \lambda_1 + \lambda_2, \; \hat{\lambda} = \lambda_1 \tag{4.8}$$

In this case the operator (4.2) will have the form (4.7), and the operator $\tilde{T}_2(\Lambda)$ in $\hat{L}_2^\Lambda(T^\nu, d\hat{\lambda})$ is written as

$$\tilde{T}_2(\Lambda)\, f(\hat{\lambda}) = a_\Lambda(\hat{\lambda})\, f(\hat{\lambda}) + \int_{T^\nu} S_\Lambda(\hat{\lambda}, \hat{\mu})\, f(\hat{\mu})\, d\hat{\mu}, \qquad f \in \hat{L}_2^\Lambda(T^\nu, d\hat{\lambda}) \tag{4.9}$$

where

$$a_\Lambda(\hat{\lambda}) = a(\hat{\lambda}) \cdot a(\Lambda - \hat{\lambda}) \tag{4.10}$$

$$S_\Lambda(\hat{\lambda}, \hat{\mu}) = S(\hat{\lambda}, \Lambda - \hat{\lambda}, \hat{\mu}, \Lambda - \hat{\mu}) \tag{4.11}$$

As is evident from Lemma 3.1 or from the representation (3.7) for the function $a(\hat{\lambda})$, there exist exactly two critical points of the function $a_\Lambda(\hat{\lambda})$

for every $\Lambda \in T^\nu$: $\widehat{\lambda_{cr}} = \Lambda/2$ and $\widehat{\lambda'_{cr}} = \Lambda/2 + \pi$ i.e., for every fixed $\Lambda = \lambda_1 + \lambda_2$ the points $\lambda_{cr} = (\lambda_1^{cr}, \lambda_2^{cr}) = (\Lambda/2, \Lambda/2)$ and $\lambda'_{cr} = (\Lambda/2 + \pi, \Lambda/2 + \pi)$ are critical.

From formula (4.9) one can see that the operator $\tilde{T}_2(\Lambda)$ (belonging to the class of the Friedrichs operators[7] for every $\Lambda \in T^\nu$ has an absolutely continuous spectrum coinciding with the range of the function $a_\Lambda(\hat{\lambda})$, $\hat{\lambda} \in T^\nu$, and possibly a finite set of eigenvalues,

$$\varepsilon_1(\Lambda), ..., \varepsilon_k(\Lambda), \qquad k = k(\Lambda) \tag{4.12}$$

which are outside of the continuous spectrum.[6,7] In addition, it can be shown[15] that there exists a neighborhood of zero $O \subset T^\nu$ such that the set of eigenvalues (4.12) is empty when $\Lambda \in O$. If we denote by

$$E = \max_{\Lambda, k} \varepsilon_k(\Lambda)$$

then

$$E < \max_{\Lambda, \hat{\lambda}} a_\Lambda(\hat{\lambda}) = a_{\Lambda = 0}(0) = a^2(0) \tag{4.13}$$

Finally, the case $\nu = 1$ the set of eigenvalues (4.12) is empty for every $\Lambda \in T^\nu$.[8,9,20]

For every $\Lambda \in T^\nu$ we denote the subspace of $\hat{L}_2^\Lambda$ on which the operator $\tilde{T}_2(\Lambda)$ has only the absolutely continuous spectrum by $\hat{L}_{2,\text{ac}}^\Lambda \subseteq \hat{L}_2^\Lambda$, and the linear span of the eigenvectors of the operator $\tilde{T}_2(\Lambda)$ by $\hat{L}_{2,\text{disc}}^\Lambda \subseteq \hat{L}_2^\Lambda$. The decomposition

$$\hat{L}_2^\Lambda = \hat{L}_{2,\text{ac}}^\Lambda \oplus \hat{L}_{2,\text{disc}}^\Lambda$$

(for every $\Lambda \in T^\nu$) generates the decomposition

$$\hat{L}_2^{\text{sym}} = \hat{L}_{2,\text{ac}}^{\text{sym}} \oplus \hat{L}_{2,\text{disc}}^{\text{sym}} \tag{4.14}$$

where

$$\hat{L}_{2,\text{ac}}^{\text{sym}} = \int \hat{L}_{2,\text{ac}}^\Lambda \, d\Lambda, \qquad \hat{L}_{2,\text{disc}}^{\text{sym}} = \int \hat{L}_{2,\text{disc}}^\Lambda \, d\Lambda$$

Let $\widetilde{T_{2,\text{ac}}}$ and $\widetilde{T_{2,\text{disc}}}$ be the parts of the operator $\tilde{T}_2$ acting respectively, in the subspaces (4.14), and we introduce the analogous designation for the operator $\widetilde{U_x^{(2)}}$.

From (4.13) it follows that the spectrum of the operator $\tilde{T}_2$ in the subspace $\hat{L}_{2,\text{disc}}^{\text{sym}}$ is separated from the upper boundary of the spectrum of $\tilde{T}_2$

in the subspace $\hat{L}^{\mathrm{sym}}_{2,\,\mathrm{ac}}$. It follows from the general scattering theory for the Friedrichs operators[6] that for every $\Lambda \in T^{\nu}$ there exists a unitary mapping (so-called "wave" operator)

$$\Omega(\Lambda)\colon\quad \hat{L}^{\Lambda}_{2,\,\mathrm{ac}} \to \hat{L}^{\Lambda}_{2}$$

transforming the operator $\tilde{T}_2(\Lambda)|_{\hat{L}^{\Lambda}_{2,\,\mathrm{ac}}}$ into the operator $T_2^{(0)}(\Lambda)$:

$$(\widetilde{T_2^{(0)}(\Lambda)}f)(\hat{\lambda}) = a_\Lambda(\hat{\lambda})f(\hat{\lambda}), \qquad f \in \hat{L}^{\Lambda}_2$$

Hence it follows that the operator

$$\Omega = \int_{T^{\nu}} \Omega(\Lambda)\, d\Lambda$$

realizes the unitary mapping

$$\Omega\colon\quad \hat{L}^{\mathrm{sym}}_{2,\,\mathrm{ac}} \to \hat{L}^{\mathrm{sym}}_{2}$$

transforming the operator $\tilde{T}_2|_{\hat{L}^{\mathrm{sym}}_{2,\,\mathrm{ac}}}$ into the operator

$$\widetilde{T_2^{(0)}}f(\lambda_1, \lambda_2) = a(\lambda_1)\, a(\lambda_2) f(\lambda_1, \lambda_2), \qquad f(\lambda_1, \lambda_2) \in \hat{L}^{\mathrm{sym}}_2$$

and the operator $\widetilde{U_x^{(2)}}|_{\hat{L}^{\mathrm{sym}}_{2,\,\mathrm{ac}}}$ into the operator (4.7).

Lemma 4.1. Let $A \subset Z^{\nu}$ be a finite set such that $|A|$ is even; then the function

$$f_A(\lambda_1, \lambda_2) = (V_2 P_{H_2} \sigma_A)(\lambda_1, \lambda_2) \in \hat{L}^{\mathrm{sym}}_2\,(T^{\nu} \times T^{\nu}, d\lambda_1\, d\lambda_2)$$

where P_{H_2} is a projection on the space H_2, has an analytic extension to the region $W_\beta \times W_\beta$.

Proof. The proof is analogous to the proof of Lemma 3.3. It can be obtained from the reasoning of refs. 5 and 10. The proof is based on the constructions of the subspace H_2 and the mapping V_2, as well as on the general estimates for cumulants of the Ising field under small β.[16]

Proof of Theorem 2. Using formula (3.10), we have that for even $|A|$ and $|B|$

$$\langle \sigma_{A+y}, \sigma_B \rangle = (T_2^{y^{(\nu+1)}+a} U_{\bar{y}}^{(2)} P_{H_2} \sigma_{A-ae_{\nu+1}}, P_{H_2}\sigma_B) + O((C\beta)^{4y^{(\nu+1)}}) \qquad (4.15)$$

In what follows it will be shown that under condition (2.8) the main contribution to the asymptotics of (4.15) is given by the first term.

Let us consider the second part of Theorem 2: $\bar{y}=0$ and v is arbitrary; then

$$(T_2^{y(v+1)+a}P_{H_2}\sigma_{A-ae_{v+1}}, P_{H_2}\sigma_B)$$
$$=\iint_{T^v\times T^v}(\tilde{T}_2^{y(v+1)+a}f_{A-ae_{v+1}})(\lambda_1,\lambda_2)\bar{f}_B(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2$$
$$=\iint_{T^v\times T^v}(\widetilde{T_{2,\mathrm{ac}}}^{y(v+1)+a}f^{\mathrm{ac}}_{A-ae_{v+1}})(\lambda_1,\lambda_2)$$
$$\times\bar{f}^{\mathrm{ac}}_B(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2+O((E)^{y(v+1)})$$
$$=\iint_{T\times T}(a(\lambda_1)\,a(\lambda_2))^{y(v+1)+a}g_{A-ae_{v+1}}(\lambda_1,\lambda_2)$$
$$\times\overline{g_B}(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2+O((E)^{y(v+1)}) \tag{4.16}$$

Here $f^{\mathrm{ac}}_{A,(B)}(\lambda_1,\lambda_2)$ is a projection of $f_{A,(B)}(\lambda_1,\lambda_2)$ on the subspace $\hat{L}^{\mathrm{sym}}_{2,\mathrm{ac}}$, and

$$g_A(\lambda_1,\lambda_2)=(\Omega f^{\mathrm{ac}}_A)(\lambda_1,\lambda_2) \tag{4.17}$$

To find the asymptotics of (4.16) we have to apply the Laplace method to the integral in (4.16) (see, for example, ref. 17) and to do this we have to know the behavior of the function $g_{A-ae_{v+1}}(\lambda_1,\lambda_2)\overline{g_B}(\lambda_1,\lambda_2)$ in the neighborhood $O\in T^v\times T^v$ of the point $0=(0,0)\in T^v\times T^v$, where the maximum of the function $a(\lambda_1)\,a(\lambda_2)$ is attained. It turns out that in O the singularities of the function $g_A(\lambda_1,\lambda_2)$ are on the manifold $\{\lambda_1=\lambda_2\}$. Below in Lemma 4.2 we shall describe these singularities.

Let us introduce local coordinates in $O\in T^v\times T^v$:

$$\Lambda=\lambda_1+\lambda_2,\qquad \zeta=\lambda_1-\hat{\lambda}_{\mathrm{cr}}=\hat{\lambda}-\hat{\lambda}_{\mathrm{cr}} \tag{4.18}$$

and let $a_\Lambda(\zeta)$ and $g_A(\Lambda,\zeta)$ be the functions $a(\lambda_1,\lambda_2)$ and $g_A(\lambda_1,\lambda_2)$ written in the coordinates (Λ,ζ) in the neighborhood $O\in T^v\times T^v$. For every fixed, sufficiently small Λ the function $a_\Lambda(\zeta)$ as a function of ζ has a unique critical point $\zeta=0$ in a small neighborhood of zero. Let $A_\Lambda(\zeta)$ be a quadratic form coinciding with the second differential of the function $d_\Lambda(\zeta)$ at this critical point. Note that $A_\Lambda(\zeta)$ is negative definite for all sufficiently small Λ; therefore $A_\Lambda(\zeta)$ is equal to zero only at the point $\zeta=0$ (i.e., on the manifold $\{\lambda_1=\lambda_2\}$).

Lemma 4.2. The function $g_A(\Lambda,\zeta)$ has the following representation when Λ and ζ are small enough:

1. For $\nu=1$

$$g_A(\Lambda,\zeta)=|A_\Lambda(\zeta)|^{1/2}\,c_1(\Lambda,\zeta)+c_2(\Lambda,\zeta) \tag{4.19}$$

where $c_k(\Lambda,\zeta)$, $k=1,2$, are analytic functions in $O\in T^\nu\times T^\nu$, and $c_2(\Lambda,0)=0$.

2. For $\nu=2$

$$g_A(\Lambda,\zeta)=\frac{c_1(\Lambda,\zeta)\ln|A_\Lambda(\zeta)|+c_2(\Lambda,\zeta)}{b_1(\Lambda,\zeta)\ln|A_\Lambda(\zeta)|+b_2(\Lambda,\zeta)}$$

where $c_k(\Lambda,\zeta)$, $b_k(\Lambda,\zeta)$, $k=1,2$, are analytic functions in $O\in T^\nu\times T^\nu$, and $c_1(\Lambda,0)=0$.

3. For odd $\nu\geqslant 3$

$$g_A(\Lambda,\zeta)=|A_\Lambda(\zeta)|^{(\nu-2)/2}\,c_1(\Lambda,\zeta)+c_2(\Lambda,\zeta)$$

where $c_k(\Lambda,\zeta)$, $k=1,2$, are analytic functions in $O\in T^\nu\times T^\nu$.

4. For even $\nu\geqslant 4$

$$g_A(\Lambda,\zeta)=\frac{c_1(\Lambda,\zeta)\,|A_\Lambda(\zeta)|^{(\nu-2)/2}\ln|A_\Lambda(\zeta)|+c_2(\Lambda,\zeta)}{b_1(\Lambda,\zeta)\,|A_\Lambda(\zeta)|^{(\nu-2)/2}\ln|A_\Lambda(\zeta)|+b_2(\Lambda,\zeta)}$$

where $c_k(\Lambda,\zeta)$, $b_k(\Lambda,\zeta)$, $k=1,2$, are analytic functions in $O\in T^\nu\times T^\nu$.

Proof. The proof is given in Appendix D.

Now from Lemma 4.2, using the Laplace method for the integral in (4.16), we obtain the formulas (2.11)–(2.13), where in the case $y_0=e_{\nu+1}$ we have $\exp[-2(m_\nu(y_0),y_n)]=(a(0,0))^{|y_n|}$.

We are coming now to the first part of the theorem, when $\nu=1$ and the first coordinate of the vector y_0 is not equal to zero: $y_0^{(1)}\neq 0$. As discussed above [see (4.15) and (4.2)], the asymptotics of the correlations $\langle\sigma_{A+y_n},\sigma_B\rangle$ amounts to finding the asymptotics of the integral

$$\iint_{T\times T}(a(\lambda_1)\,a(\lambda_2))^{y_n^{(2)}}\exp[iy_n^{(1)}(\lambda_1+\lambda_2)]$$
$$\times g_{A-ae_2}(\lambda_1,\lambda_2)\,\overline{g_B}(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2 \tag{4.20}$$

as $y_n=(y_n^{(1)},y_n^{(2)})\to\infty$, $n\to\infty$ along the vector $y_0=(y_0^{(1)},y_0^{(2)})$.

As for odd $|A|$ and $|B|$, in this case we can use the saddle-point method. Let us deform the contour $T^1\times T^1$ to a contour $\Gamma\times\Gamma$, where $\Gamma=\Gamma(s_0)=(\lambda+is_0,\lambda\in T^1)$, and $is_0=\lambda_0=\lambda_0(\xi)$ is the critical point of the function $\ln a(\lambda)+i\xi\lambda$, $\xi=y_0^{(1)}/y_0^{(2)}$, discussed in the previous section. Let

$R \subset W_\beta \times W_\beta$ be a subset of the region $W_\beta \times W_\beta$ formed by all contours $\Gamma(s) \times \Gamma(x)$ such that $-G_1 < s < G_1$; see (3.4). To use the saddle-point method we have to construct an extension of the functions $g_{A,(B)}(\lambda_1, \lambda_2)$ defined on $T^1 \times T^1$ (i.e., for $s = 0$) to the set R, so that the integral (4.20) is equal to the following integral

$$\iint_{\Gamma(s_0) \times \Gamma(s_0)} (a(\lambda_1)\, a(\lambda_2))^{y_n^{(2)}} \exp[iy_n^{(1)} (\lambda_1 + \lambda_2)]$$

$$\times g_{A - ae_2}(\lambda_1, \lambda_2)\, \overline{g_B}(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2$$

$$= \iint_{\Gamma(s_0) \times \Gamma(s_0)} \exp\{ y_n^{(2)} (\ln a(\lambda_1) + i\xi\lambda_1 + \ln a(\lambda_2) + i\xi\lambda_2)\}$$

$$\times g_{A - ae_2}(\lambda_1, \lambda_2)\, \overline{g_B}(\lambda_1, \lambda_2)\, d\lambda_1\, d\lambda_2 \tag{4.21}$$

We can conveniently introduce in R the coordinates

$$\Lambda = \lambda_1 + \lambda_2, \qquad \lambda = \operatorname{Re} \lambda_1 \tag{4.22}$$

which are a generalization of the coordinates (4.8) in $T^1 \times T^1 \subset R$, and let $a_\Lambda(\lambda)$ be the function $a(\lambda_1, \lambda_2)$ written in these coordinates. It is evident that Λ passes by the complex manifold

$$W_{2\beta} = \{\Lambda \colon \operatorname{Re} \Lambda \in T^1,\ |\operatorname{Im} \Lambda| < 2G_1\}$$

and λ runs through the torus T^1. We recall that for every real $\Lambda \in T^1$ there exist two critical points of the function $a_\Lambda(\lambda) = a_\Lambda(\hat{\lambda})$: $\lambda_{\text{cr}} = \lambda_{\text{cr}}(\Lambda) = \widehat{\lambda_{\text{cr}}} \in T^1$ and $\lambda'_{\text{cr}} = \lambda'_{\text{cr}}(\Lambda) = \widehat{\lambda'_{\text{cr}}} \in T^1$, which are on the manifold $\{\lambda_1 = \lambda_2\}$, and the analogous critical points of the function $a_\Lambda(\lambda)$ exist for every fixed $\Lambda \in W_{2\beta}$. If we denote small neighborhoods of these critical points by O_1 and O_2, respectively, then let

$$\Lambda = \lambda_1 + \lambda_2, \qquad \zeta = \lambda - \lambda_{\text{cr}} \tag{4.23}$$

be local coordinates in O_1 (and we have analogous ones in O_2).

Lemma 4.3. For any even $|A|$ the function $g_A(\lambda_1, \lambda_2) = g_A(\Lambda, \lambda)$, defined by the formula (4.17) for $\lambda_1, \lambda_2 \in T^1$ and written in the coordinates (4.22) has a continuous extension to the set R such that:

1. The function $g_A(\Lambda, \lambda)$ is analytic with respect to $\Lambda \in W_{2\beta}$ for every fixed $\lambda \in T^1$.

2. For every fixed $\Lambda \in W_{2\beta}$ the function $g_A(\Lambda, \lambda)$ is real-analytic with respect to $\lambda \in T^1$, except for the critical-points $\lambda_{\text{cr}}(\Lambda)$, and $\lambda'_{\text{cr}}(\Lambda)$, and in

the neighborhoods O_1 and O_2 of these points the function $g_A(A, \lambda)$ has the following representation [in the local coordinates (4.23)]:

$$g_A(A, \zeta) = |\zeta|\, c_1^{(1)}(A, \zeta) + c_2^{(1)}(A, \zeta) \quad \text{in} \quad O_1 \tag{4.24}$$

and we have the same representation in O_2. Here the functions $c_k^{(i)}(A, \zeta)$, $i, k = 1, 2$, are analytic of $A \in W_{2\beta}$ and real-analytic of ζ in a small neighborhood of the point $\zeta = 0$. In addition,

$$c_2^{(i)}(A, 0) = 0, \qquad i = 1, 2 \tag{4.25}$$

Proof. The proof is given in Appendix E.

Now the equality of the integrals (4.20) and (4.21) follows from Lemma 4.3. Further, as is seen from the reasoning of the previous section, the maximum of the real part of the function

$$f(\lambda_1, \lambda_2, \xi) = \ln a(\lambda_1) + i\xi\lambda_1 + \ln a(\lambda_2) + i\xi\lambda_2$$

on the contour $\Gamma(s_0) \times \Gamma(s_0)$ is attained at the point $\lambda_1 = \lambda_2 = is_0 = \lambda_0$, and this contour is tangent to the level line of the imaginary part of the function $f(\lambda_1, \lambda_2, \xi)$. Hence the neighborhood of the point

$$P_0 = (\lambda_0, \lambda_0) \in \Gamma(s_0) \times \Gamma(s_0) \subset R$$

makes the main contribution to the asymptotics. Since the second differential of the function $f(\lambda_1, \lambda_2, \xi)$ is nondegenerate at the point P_0, we can apply standard practice to calculate the asymptotics of (4.21) using the Laplace method. Taking into account the character of the singularities of the functions $g_A(A, \zeta)$ and $\overline{g_B}(A, \zeta)$ at the point P_0 (afforded by Lemma 4.3), we obtain the formula (2.11).

Theorem 2 is proved.

APPENDIX A. PROOF OF LEMMA 3.1

The proof is based on a construction of the space H_1, as well as on some sharp estimates used in this construction (given in refs. 5, 11, and 12). Here we recall the main steps of the construction of H_1 and some implications of the sharp estimates which are necessary for the deduction of the representations (3.5) and (3.6).

A1. The Multiplicative Basis in *H*

For every point $x \in Y_0 \subset Z^{\nu+1}$ of the zero-slice Y_0 we denote the subset $\{y \in Y_0 : y < x\}$ by $V_x \subset Y_0$, where $y < x$ in terms of the usual lexicographic ordering on $Y_0\,(=Z^\nu)$. Let μ_β be a distribution of the Ising field on the lattice $Z^{\nu+1}$, and for every configuration $\sigma\in\{-1,1\}^{Z^{\nu+1}}$ we define

$$\tau_x(\sigma)=\tau_x(\sigma\,|_{V_x})=\langle\sigma(x)\;|\sigma|_{V_x}\rangle_{\mu_\beta}$$

where $\langle\cdot\,|\sigma|_{V_x}\rangle_{\mu_\beta}$ is a conditional average under the condition that the values of the configuration σ on the set V_x are fixed (and coincide with $\sigma|_{V_x}$).

Let us introduce functions

and
$$u_x(\sigma)=\frac{\sigma(x)-\tau_x(\sigma)}{(1-\tau_x^2(\sigma))^{1/2}}\in H,\qquad x\in Y_0$$

$$u_I(\sigma)=\prod_{x\in I}u_x(\sigma)$$

for any finite subset $I\subset Y_0$. It turns out that the set of the functions $\{u_I, I\subset Y_0\}$ forms an orthonormal basis in the space H. In addition, the following expansion is valid:

$$u_x(\sigma)=\sigma(x)-\beta\sum_{k=1,\dots,\nu}\sigma(x-e_k)+\widetilde{u}_x$$

where e_k is a unit vector in Z^ν which has the direction of the kth axis, and $\widetilde{u}_x$ has the following representation:

$$\widetilde{u}_x=\sum_{I\subset V_x\cup\{x\}}B_I^x\,\sigma_I$$

with coefficients B_I^x. From the sharp estimates of the coefficients B_I^x given in ref. 5, it follows that for any $x, y\in Y_0, x\leqslant y$,

$$(Tu_x, u_y)=\left\langle\sigma(x+e_{\nu+1})\left[\sigma(y)-\beta\sum_{k=1,\dots,\nu}\sigma(y-e_k)\right]\right\rangle + O((C\beta)^{|x-y|+2})\tag{A.1}$$

where $|\xi|=\sum_{i=1,\dots,\nu}|\xi^{(i)}|$ for $\xi=(\xi^{(1)},\dots,\xi^{(\nu)})\in Z^\nu$. The analogous representation is valid for $x\geqslant y$.

A2. The Space H_1 and the Basis in H_1

The invariant subspace H_1 mentioned in Section 3 is constructed in ref. 5 as a small perturbation of the space $H_1^0\subset H$, where H_1^0 is the linear span of the vectors $\{u_x, x \in Y_0\}$. In this case there exists in H_1 an orthonormal basis of the form

$$v_x = u_x + \sum_{I \subset Y_0, |I| \geqslant 2} S_I^x u_I$$

where S_I^x are coefficients. From the estimates of these coefficients given in ref. 5, it follows that the matrix elements

$$a_{x-y} = (T_1 v_x, v_y)_{H_1} \tag{A.2}$$

of the operator $T_1 = T|_{H_1}$ in the basis $\{v_x, x \in Y_0\}$ can be represented as

$$(T_1 v_x, v_y)_{H_1} = (T_1 u_x, u_y)_{H_1} + O((C\beta)^{|x-y|+2}) \tag{A.3}$$

Further, for any $y = (y^{(1)},..., y^{(\nu)}) \in Y_0$ we have

$$\langle \sigma(e_{\nu+1})\, \sigma(y)\rangle = \beta^{|y|+1} \frac{(|y|+1)!}{\prod_{i=1,..., \nu} |y^{(i)}|!} + O((C\beta)^{|y|+2}) \tag{A.4}$$

Formula (A.4) follows from the well-known formula for the average $\langle F\rangle_{\mu_\beta}$ with respect to the distribution μ_β:

$$\langle F\rangle_{\mu_\beta} = \sum_{n=0} \frac{\beta^n}{n!} \sum_{(b_1,..., b_n)} \langle F, \sigma_{b_1},..., \sigma_{b_n}\rangle_0$$

where the summation is over all ordered sets $(b_1,..., b_n)$ of unoriented links of the lattice $Z^{\nu+1}$, $\sigma_b = \sigma_x \cdot \sigma_y$, where $b = (x, y)$, and $\langle \cdot,..., \cdot\rangle_0$ is a cumulant calculated with respect to the distribution of probabilities of the nonperturbed field, with independent values distributed by the probabilities

$$\Pr(\sigma(x) = 1) = \Pr(\sigma(x) = -1) = 1/2$$

at every point $x \in Z^{\nu+1}$. From (A.1)–(A.4) we have that

$$a_u = \beta^{|u|+1} \frac{|u|!}{\prod_{i=1,..., \nu} |u^{(i)}|!} + O((C\beta)^{|u|+2}) \tag{A.5}$$

where $u = (u^{(1)},..., u^{(\nu)})$.

A3. The Mapping V_1

The unitary mapping $V_1: H_1 \to L_2(T^\nu, d\lambda)$ is given by the formula

$$V_1(v_x) = e^{i(\lambda, x)} \in L_2(T^\nu, d\lambda)$$

In so doing, the operator T_1 with the matrix elements (A.2) in the basis $\{v_x, x \in Y_0\}$ is transformed into the operator of multiplication by the function

$$\tilde{a}(\lambda) = \sum_{u \in Z^\nu} a_u e^{i(u, \lambda)}$$

Now the conclusion of the lemma follows from (A.5).

APPENDIX B. PROOF OF LEMMA 3.2

1. In the case $\nu = 1$ the function $a(\lambda, \beta)$ can be written as

$$\begin{aligned} a(\lambda, \beta) &= \beta \left(1 + \frac{\beta e^{i\lambda}}{1 - \beta e^{i\lambda}} + \frac{\beta e^{-i\lambda}}{1 - \beta e^{-i\lambda}} \right) + \beta^2 a_1(\lambda, \beta) \\ &= \frac{\beta}{(1 - \beta e^{i\lambda})(1 - \beta e^{-i\lambda})} + \beta^2 a_2(\lambda, \beta) \end{aligned} \tag{B.1}$$

where the functions $a_k(\lambda, \beta)$, $k = 1, 2$, are uniformly bounded inside the region W_β:

$$|a_k(\lambda, \beta)| \leqslant C_k$$

Then taking into account the inequality

$$\left| \frac{\beta}{(1 - \beta e^{i\lambda})(1 - \beta e^{-i\lambda})} \right| \geqslant \frac{\beta}{(1 + 1/D_1)^2} > 0$$

where D_1 is the constant defined in (3.4), we get that $a(\lambda, \beta) \neq 0$ in W_β.

In the case $\nu = 2$ we shall separate the region W_β into four covering subregions:

$$\begin{aligned} W_1 &= \{ |\mathrm{Im}\, \lambda^{(k)}| \leqslant \tfrac{1}{2} |\ln \beta| + \delta, k = 1, 2\} \\ W_2 &= \{ |\mathrm{Im}\, \lambda^{(1)}| \leqslant \tfrac{1}{2} |\ln \beta| + \delta, |\mathrm{Im}\, \lambda^{(2)}| \geqslant \tfrac{1}{2} |\ln \beta| \} \\ W_3 &= \{ |\mathrm{Im}\, \lambda^{(2)}| \leqslant \tfrac{1}{2} |\ln \beta| + \delta, |\mathrm{Im}\, \lambda^{(1)}| \geqslant \tfrac{1}{2} |\ln \beta| \} \\ W_4 &= \{ |\mathrm{Im}\, \lambda^{(k)}| \geqslant \tfrac{1}{2} |\ln \beta|, k = 1, 2\} \end{aligned}$$

where δ is a small constant. Note that by the evenness of the function $a_0(\lambda, \beta)$ [see (3.6)] it is sufficient to consider the case when $\mathrm{Im}\, \lambda^{(k)} > 0$, $k = 1, 2$. We denote

$$W_\beta^+ = W_\beta \cap \{ \mathrm{Im}\, \lambda^{(k)} > 0, k = 1, 2\}$$

If $\lambda \in W_\beta^+ \cap W_1$, then $|a_0(\lambda, \beta)| \geqslant (1 - k_1\sqrt{\beta})$, $k_1 > 0$ is a constant, and the function $a(\lambda, \beta)$ has an analogous estimate.

If $\lambda \in W_\beta^+ \cap W_2$, then

$$a_0(\lambda, \beta) = 1 + \frac{\beta e^{-i\lambda^{(2)}}}{1 - \beta e^{-i\lambda^{(2)}}} + b(\lambda, \beta)$$

where $|b(\lambda, \beta)| \leqslant k_2\sqrt{\beta}$, when $\lambda \in W_\beta^+ \cap W_2$, k_2 is a constant. Hence

$$|a_0(\lambda, \beta)| \geqslant \left|1 + \frac{\beta e^{-i\lambda^{(2)}}}{1 - \beta e^{-i\lambda^{(2)}}}\right| - k_2\sqrt{\beta} \geqslant \left(\frac{1}{1 + 1/D_2} - k_2\sqrt{\beta}\right) \geqslant k_3$$

where $k_3 > 0$ is a constant, and the function $a(\lambda, \beta)$ has an analogous estimate in the region $W_\beta^+ \cap W_2$. The case when $\lambda \in W_\beta^+ \cap W_3$ is considered in a similar way.

If $\lambda \in W_\beta^+ \cap W_4$, then

$$a_0(\lambda, \beta) = 1 + \frac{\beta(e^{-i\lambda^{(1)}} + e^{-i\lambda^{(2)}})}{1 - \beta(e^{-i\lambda^{(1)}} + e^{-i\lambda^{(2)}})} + \tilde{b}(\lambda, \beta)$$

where $|\tilde{b}(\lambda, \beta)| \leqslant k_4\beta^{3/2}$ when $\lambda \in W_\beta^+ \cap W_4$. Hence

$$|a_0(\lambda, \beta)| \geqslant \left(\frac{1}{1 + 2/D_2} - k_4\beta^{3/2}\right) \geqslant k_5$$

where $k_4, k_5 > 0$ are constants, and the function $a(\lambda, \beta)$ has an analogous estimate in this region. Thus, $a(\lambda, \beta) \neq 0$ inside the region W_β. The cases of the other dimensions $\nu \geqslant 3$ can be considered along similar lines.

2. Let $\lambda = ix$, $x \in R^\nu$. From the Fourier-series expansion of the function $a(\lambda, \beta)$,

$$a(\lambda, \beta) = \sum_{\bar{n}} b_{\bar{n}} e^{i(\bar{n}, \lambda)}, \qquad \bar{n} = (n_1, ..., n_\nu)$$

with real coefficients $b_{\bar{n}}$ it follows that the function $a(\lambda, \beta)$ is real when $\lambda = ix$. To prove the nondegeneracy of the second differential of the function $a(\lambda, \beta)$ for pure imaginary λ it is sufficient to prove this fact for the function $a_0(\lambda, \beta)$. The second differential of

$$\exp\left\{-\sum_{k=1}^{\nu} n_k x^{(k)}\right\} \tag{B.2}$$

at the point $x_0=(x_0^{(1)},..., x_0^{(\nu)})$ is equal to

$$\left(\sum_{k=1}^{\nu} n_k\, dx^{(k)}\right)^2 \exp\left\{-\sum_{k=1}^{\nu} n_k x_0^{(k)}\right\}$$

and it is a nonnegative degenerate quadratic form, which is equal to zero on the plane

$$\sum n_k\, dx^{(k)}=0$$

But the intersection of all these planes contains the unique point $dx^{(1)}= \cdots =dx^{(\nu)}=0$ when $\{n_k\}$ runs through the integral lattice, and every exponent (B.2) is a part of the sum (3.6) with the positive coefficients

$$C_{\{n_k\}}=\beta^{\sum |n_k|}\frac{(\sum |n_k|)!}{\prod |n_k|!}>0$$

(recall that we consider the ferromagnetic case when $\beta>0$). Consequently, the total quadratic form corresponding to the second differential of the function $a_0(\lambda, \beta)$ with pure imaginary $\lambda_0=ix_0$ is positive,

$$\sum_{\{n_k\}} C_{\{n_k\}}\left(\sum_{k=1}^{\nu} n_k\, dx^{(k)}\right)^2 \exp\left\{-\sum_{k=1}^{\nu} n_k x_0^{(k)}\right\}>0$$

for every nonzero $dx=(dx^{(1)},..., dx^{(\nu)})$.

3. The conclusion of the lemma follows from the representation (B.1) and also (3.7) for the function $a(\lambda, \beta)$.

The lemma is proved.

APPENDIX C. PROOF OF LEMMA 3.4

First we consider the case when $\nu=1$, and let $\xi^{(1)}=\xi>0$. We can find the critical points of the function $f_\xi(\lambda)$ as the solutions of the equation

$$\frac{d}{d\lambda} f_\xi(\lambda)=\frac{a'(\lambda)}{a(\lambda)}+i\xi=0, \qquad \lambda\in W_\beta$$

Taking into account the representation (B.1) of the function $a(\lambda)$ and using Rousher's theorem, we can prove the existence of two critical points of $f_\xi(\lambda)$:

$$\lambda_0^{(\pm)}(\xi, \beta)= -i\ln\frac{\pm[\xi^2(1-\beta^2)^2+4\beta^2]^{1/2}-\xi(1+\beta^2)}{2\beta(1-\xi)}+\alpha^{(\pm)}(\xi, \beta)$$

$\operatorname{Im}\lambda_0^{(+)}>0$, $\operatorname{Im}\lambda_0^{(-)}<0$, $|\alpha^{(\pm)}(\xi, \beta)|<C\beta$, C is an absolute constant.

If $\xi \geqslant \beta$, then the critical points $\lambda_0^{(\pm)}(\xi, \beta)$ will lie in the region $|\mathrm{Im}\,\lambda| \geqslant \frac{1}{2}|\ln \beta| + \delta$, where $\delta > 0$ is a constant. In particular, for ξ such that $\xi \geqslant \alpha > 0$, α a is a constant, it is easy to see that the critical points have the following representation:

$$\lambda_0^{(+)}(\xi, \beta) = -i\left(\ln \beta + \ln\left(1 + \frac{1}{\xi}\right) + \alpha^{(+)}(\xi, \beta)\right), \qquad \mathrm{Im}\,\lambda_0^{(+)} > 0$$

$$\lambda_0^{(-)}(\xi, \beta) = \pi + i\left(\ln \beta + \ln\left(\frac{1}{\xi} - 1\right) + \alpha^{(-)}(\xi, \beta)\right), \quad 0 < \xi < 1, \ \mathrm{Im}\,\lambda_0^{(-)} < 0$$

where $|\alpha^{(\pm)}(\xi, \beta)| < C\beta$, C is an absolute constant.

Since the critical points $\lambda_0^{(\pm)}(\xi, \beta)$ must be inside the region W_β, it is necessary that

$$1 + \frac{1}{\xi} > D_1 \quad \text{for } \lambda_0^{(+)} \in W_\beta, \qquad \frac{1}{\xi} - 1 > D_1 \quad \text{for } \lambda_0^{(-)} \in W_\beta$$

where D_1 is the constant from (3.4). But to use the saddle-point method only the critical point $\lambda_0^{(+)}(\xi, \beta)$ with pure imaginary coordinates will be important for us (as will be shown below), so we shall restrict our consideration to the first inequality, which leads to the following estimate on ξ:

$$\xi < \frac{1}{D_1 - 1}$$

To find the saddle-point contour, we have to consider the level line of the function $\mathrm{Im}\, f_\xi(\lambda)$ passing through the points $\lambda_0^{(+)}(\xi, \beta)$ and $\lambda_0^{(-)}(\xi, \beta)$. A detailed analysis of the function $\mathrm{Re}\, f_\xi(\lambda)$ shown that the saddle-point contour goes through the point $\lambda_0^{(+)}$ parallel to the torus T. As for tile other critical point $\lambda_0^{(-)}$, the corresponding contour ought to have the vertical tangent at the point $\lambda_0^{(-)}$. But for any such contour there always exists a point $\lambda' \neq \lambda_0^{(-)}$ on this contour such that

$$\mathrm{Re}\, f_\xi(\lambda') > \mathrm{Re}\, f_\xi(\lambda_0^{(-)})$$

Hence there does not exist a saddle-point contour passing through the point $\lambda_0^{(-)}$.

For arbitrary real ξ such that $|\xi| \geqslant \alpha > 0$, in the case $\nu = 1$ under the condition

$$|\xi| < \frac{1}{D_1 - 1}$$

there exists the unique saddle point for the integral (3.15) inside the region W_β, which is equal to

$$\lambda_0(\xi, \beta) = -i \operatorname{sign} \xi \cdot \left(\ln \beta + \ln\left(1 + \frac{1}{|\xi|}\right) + \alpha(\xi, \beta)\right)$$

where $|\alpha(\xi, \beta)| < C\beta$, C is an absolute constant.

For small $0 < \xi < \sqrt{\beta}$ the corresponding critical points $\lambda_0^{(\pm)}(\xi, \beta)$ are inside the region $|\operatorname{Im} \lambda| \leqslant \frac{1}{2} |\ln \beta| + \delta'$, $\delta' > \delta$, and only one of them, $\lambda_0^{(+)}(\xi, \beta)$, with pure imaginary coordinate is the saddle point for the integral (3.15).

In the case $\nu > 1$ we can divide the region W_β into 2^ν subregions in the same way as in Appendix B. Then we can prove the existence of a unique critical point with pure imaginary coordinates in each of these subregions using the representations (3.5) and (3.6) for the function $a(\lambda, \beta)$. In so doing, each of these subregions corresponds to some values of the parameters $\xi = (\xi^{(1)},..., \xi^{(\nu)})$ as we have explained above. If

$$\min_{k=1,...,\nu} |\xi^{(k)}| \geqslant \alpha > 0$$

where α is an absolute constant, then similar to the case $\nu = 1$, under the conditions

$$\left(1 + \sum_{s=1,...,\nu} |\xi^{(s)}|\right) > D_\nu |\xi^{(k)}|, \qquad k = 1,..., \nu$$

in W_β there exists a unique saddle point $\lambda_0(\xi, \beta)$ for the integral (3.15) with pure imaginary coordinates

$$\lambda_0^{(k)}(\xi, \beta) = -i \operatorname{sign} \xi^{(k)} \cdot \left(\ln \beta + \ln \frac{1 + \sum_{s=1,...,\nu} |\xi^{(s)}|}{|\xi^{(k)}|} + \alpha_k(\xi, \beta)\right), \quad k = 1,..., \nu$$

where $|\alpha_k(\xi, \beta)| < C_k \beta$.

Further, the saddle-point surface is constructed in much the same way as in the case $\nu = 1$: it must pass through the point $\lambda_0(\xi, \beta)$ parallel to the torus T^ν. The lemma is proved.

APPENDIX D

Statements. 1. From the results of refs. 8, 9, and 20 it follows that in the case $\nu = 1$ for every $\Lambda \in T$

$$\Delta_\Lambda(z + i0) \neq 0$$

when $z \in \{a_\Lambda(\hat{\lambda}), \hat{\lambda} \in T\}$, and the function $a_\Lambda(\hat{\lambda})$ is defined by (4.10).

2. From Lemma 3.1, the representations (3.5)–(3.7), and the results of ref. 15, it follows that in the case $\nu \geqslant 2$ for $\Lambda \in O(0)$

$$\Delta_\Lambda(z+i0) \neq 0$$

when $z \in \{a_\Lambda(\hat{\lambda}), \hat{\lambda} \in T^\nu\}$, and the function $a_\Lambda(\hat{\lambda})$ is defined by (4.10).

Proof of Lemma 4.2. Using the explicit representation for the kernel of the wave operator $\Omega(\Lambda, \hat{\lambda}, \hat{\mu})$,[6] we can write the function $g_A(\Lambda, \hat{\lambda})$ in the notation of formulas (4.9)–(4.11) as

$$\begin{aligned} g_A(\Lambda, \hat{\lambda}) &= \int_{T^\nu} \Omega^*(\Lambda, \hat{\lambda}, \hat{\mu}) f_A^{\mathrm{ac}}(\Lambda, \hat{\mu})\, d\hat{\mu} \\ &= f_A^{\mathrm{ac}}(\Lambda, \hat{\lambda}) - \int_{T^\nu} \frac{T_\Lambda(\hat{\mu}, \hat{\lambda}, a_\Lambda(\hat{\lambda}) + i0)}{(a_\Lambda(\hat{\mu}) - a_\Lambda(\hat{\lambda}) - i0)} f_A^{\mathrm{ac}}(\Lambda, \hat{\mu})\, d\hat{\mu} \\ &= f_A^{\mathrm{ac}}(\Lambda, \hat{\lambda}) - \lim_{z \to a_\Lambda(\hat{\lambda}) + i0} \int_{T^\nu} \frac{T_\Lambda(\hat{\mu}, \hat{\lambda}, z)}{(a_\Lambda(\hat{\mu}) - z)} f_A^{\mathrm{ac}}(\Lambda, \hat{\mu})\, d\hat{\mu} \qquad \text{(D.1)} \end{aligned}$$

where

$$T_\Lambda(\hat{\mu}, \hat{\lambda}, z) = \frac{D_\Lambda(\hat{\mu}, \hat{\lambda}, z)}{\Delta_\Lambda(z)}, \qquad z \in C^1 \backslash \Theta_\Lambda \qquad \text{(D.2)}$$

Θ_Λ is the range of the function $a_\Lambda(\hat{\lambda})$, $\hat{\lambda} \in T^\nu$, $\Delta_\Lambda(z)$ is the Fredholm determinant, and $D_\Lambda(\hat{\lambda}, \hat{\mu}, z)$ is the Fredholm minor for the kernel $S_\Lambda(\hat{\lambda}, \hat{\mu})$; see (4.11) and (4.4).

Let us consider the case $\nu = 1$. Recall that the Fredholm minor and determinant can be represented in the form of series:

$$\Delta_\Lambda(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} \int_T \cdots \int_T \frac{\Delta_\Lambda^{(n)}(\hat{\xi}_1,..., \hat{\xi}_n)}{\prod_{i=1,...,n}(a_\Lambda(\hat{\xi}_i) - z)} \prod_{i=1}^{n} d\hat{\xi}_i \qquad \text{(D.3)}$$

$$D_\Lambda(\hat{\lambda}, \hat{\mu}, z) = S_\Lambda(\hat{\lambda}, \hat{\mu}) + \sum_{n=1}^{\infty} \frac{1}{n!} \int_T \cdots \int_T \frac{D_\Lambda^{(n)}(\hat{\lambda}, \hat{\mu}, \hat{\xi}_1,..., \hat{\xi}_n)}{\prod_{i=1,...,n}(a_\Lambda(\hat{\xi}_i) - z)} \prod_{i=1}^{n} d\hat{\xi}_i \qquad \text{(D.4)}$$

where $\Delta_\Lambda^{(n)}(\hat{\xi}_1,..., \hat{\xi}_n) = \det\{S_\Lambda(\hat{\xi}_i, \hat{\xi}_j)\}$; $D_\Lambda^{(n)}(\hat{\lambda}, \hat{\mu}, \hat{\xi}_1,..., \hat{\xi}_n)$ are so-called nth Fredholm minors corresponding to the kernel $S_\Lambda(\hat{\lambda}, \hat{\mu})$ (see, for example, ref. 19). It is easy to see from Hadamard's inequality that the functions $\Delta_\Lambda(z)$ and $D_\Lambda(\hat{\lambda}, \hat{\mu}, z)$ for fixed $\hat{\lambda}, \hat{\mu} \in T$ are analytic with respect to (z, Λ), when $z \notin \Theta_\Lambda$. In addition for every $x \in \Theta_\Lambda$ except maybe the critical values of the function $a_\Lambda(\hat{\lambda})$ there exist limits

$$\Delta_\Lambda^{\pm}(x) = \lim_{z \to x \pm i0} \Delta_\Lambda(z), \qquad x \in \Theta_\Lambda$$

when z tends to the upper (+) or lower (−) extent of Θ_Λ. By analogy, the function

$$I_\Lambda(\hat{\lambda}, z) = \int_T \frac{D_\Lambda(\hat{\mu}, \hat{\lambda}, z)}{(a_\Lambda(\hat{\mu}) - z)} f_\Lambda^{\mathrm{ac}}(\Lambda, \hat{\mu})\, d\hat{\mu}$$

is analytic with respect to (z, Λ) as $z \notin \Theta_\Lambda$, and it has limits, when z tends to the extremes of Θ_Λ,

$$I_\Lambda^\pm(\hat{\lambda}, x) = \lim_{z \to x \pm i0} I_\Lambda(\hat{\lambda}, z), \qquad x \in \Theta_\Lambda$$

[except maybe the critical values of $a_\Lambda(\hat{\lambda})$]. In this case the functions $\Delta_\Lambda^\pm(x)$, $I_\Lambda^\pm(\hat{\lambda}, x)$ are analytic with respect to Λ and $x \in \Theta_\Lambda$ [except the critical values of $a_\Lambda(\hat{\lambda})$].

In the local coordinates (Λ, ζ) [see (4.18)] in the neighborhood O of the point $(0, 0) \in T^1 \times T^1$ we have exactly a unique critical point λ_0 with coordinates $\lambda_0 = (\Lambda, 0)$, and it is easy to show that the following representations are valid:

$$\Delta_\Lambda^+(a_\Lambda(\zeta)) = -2\pi i \frac{S_\Lambda(0)}{|A_\Lambda(\zeta)|^{1/2}} (1 + N_\Lambda(\zeta))$$

$$I_\Lambda^+(\zeta, a_\Lambda(\zeta)) = -2\pi i \frac{S_\Lambda(0) f_\Lambda^{\mathrm{ac}}(\Lambda, 0)}{|A_\Lambda(\zeta)|^{1/2}} (1 + P_\Lambda(\zeta))$$

where $A_\Lambda(\zeta)$ is the second differential of the function $a_\Lambda(\zeta)$ at the critical point $\lambda_0 = (\Lambda, 0)$,

$$S_\Lambda(0) = S_\Lambda(0, 0) + \sum_{n=2}^{\infty} \frac{1}{(n-1)!} \int_T \cdots \int_T \frac{S_\Lambda^{(n)}(\hat{\xi}_2, \ldots, \hat{\xi}_n)}{\prod_{i=2,\ldots,n}(a_\Lambda(\hat{\xi}_i) - a_\Lambda(0))} \prod_{i=2}^{n} d\hat{\xi}_i$$

$$S_\Lambda^{(n)}(\hat{\xi}_2, \ldots, \hat{\xi}_n) = \det\{S_\Lambda(\hat{\xi}_i, \hat{\xi}_j)\}_{i,j=1}^{n} \big|_{\xi_1 = 0}$$

the functions $N_\Lambda(\zeta)$, $P_\Lambda(\zeta)$ for any $\zeta \in O(0)$ are analytic in Λ, and satisfy there the estimates

$$|N_\Lambda(\zeta)| < \mathrm{const} \cdot |\zeta|, \qquad |P_\Lambda(\zeta)| < \mathrm{const} \cdot |\zeta|, \qquad \zeta \in O(0)$$

In addition, a detailed analysis of the integrals from (D.3) and (D.4) shows that

$$\frac{I_\Lambda^+(\zeta, a_\Lambda(\zeta))}{\Delta_\Lambda^+(a_\Lambda(\zeta))} = f_\Lambda^{\mathrm{ac}}(\Lambda, 0) + R_\Lambda(\zeta)$$

where the function $R_\Lambda(\zeta)$ has the representation

$$R_\Lambda(\zeta) = |A_\Lambda(\zeta)|^{1/2} c_1(\Lambda, \zeta) + c_2(\Lambda, \zeta), \qquad \zeta \in O(0) \tag{D.5}$$

$c_j(\Lambda, \zeta), j = 1, 2$, are functions analytic in O, and $c_2(\Lambda, 0) = 0$.

Finally using the expansion of the function $f_\Lambda(\zeta)$ at the point $\zeta = 0$, from (D.1), (D.2), and (D.5) we get the representation (4.19).

The cases $\nu \geqslant 2$ are considered in a similar way. Lemma 4.2 is proved.

APPENDIX E. PROOF OF LEMMA 4.3

Let us consider in the region

$$R = \{(\lambda_1, \lambda_2) \in W_\beta \times W_\beta\colon \operatorname{Im} \lambda_1 = \operatorname{Im} \lambda_2, |\operatorname{Im} \lambda_j| < G_1, j = 1, 2\}$$

for any $\Lambda = \hat{\Lambda} + i2s$, $-\pi < \hat{\Lambda} \leqslant \pi$, $-G_1 < s < G_1$, the following manifold:

$$\Gamma_\Lambda = \{(\lambda_1, \lambda_2) \in R\colon \lambda_1 + \lambda_2 = \Lambda\}$$

$\Gamma_\Lambda \subset R$, and let $\lambda = \operatorname{Re} \lambda_1 = \operatorname{Re} \hat{\lambda}$ be a coordinate of the point situated on Γ_Λ. A projection of any manifold Γ_Λ on the cut $\{\operatorname{Im} \lambda_1 = \operatorname{Im} \lambda_2 = 0\} \subset R$ has the shape shown in Fig. 1.

As discussed above in the proof of Lemma 4.2, the function $g_\Lambda(\Lambda, \lambda)$ is analytic in Λ when $\Lambda \in T^1$ is real (i.e., for $s = 0$), real-analytic in $\lambda \in T^1$,

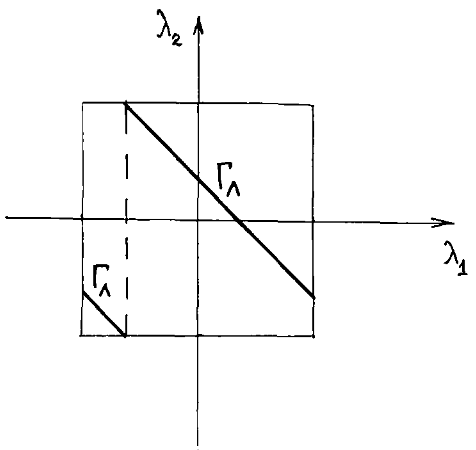

Fig. 1.

except the critical points $\lambda_{cr}(\Lambda), \lambda'_{cr}(\Lambda)$, and for every $\Lambda \in T^1$ has the representation (4.19) in the neighborhood of the critical points $\lambda_{cr} = \Lambda/2$ and $\lambda'_{cr} = \Lambda/2 + \pi$ of the function $a_\Lambda(\lambda)$.

Note that for complex Λ the values of the function $a_\Lambda(\lambda), \lambda \in T^1$, fill out a smooth curve $\gamma_\Lambda \subset C^1$ on the complex plane. Since the function $a_\Lambda(\lambda)$ has the same values at the points λ and $\operatorname{Re} \Lambda - \lambda$, the curve γ_Λ is covered twice under changes of λ from $-\pi$ to π, and the extreme points $z_1(\Lambda)$ and $z_2(\Lambda)$ of γ_Λ correspond to the critical values of the function $a_\Lambda(\lambda)$. From Lemma 3.1 or from the representation (3.7) it follows that for all $\Lambda = \operatorname{Re} \Lambda + i2s$ the function $a_\Lambda(\lambda)$ has exactly two nondegenerate critical points: λ_{cr} and λ'_{cr}.

We define now the function $g_\Lambda(\Lambda, \lambda)$ for complex $\Lambda = \operatorname{Re} \Lambda + i2s$, $-\pi < \operatorname{Re} \Lambda \leqslant \pi$, $-G_1 < s < G_1$, by the formula

$$g_\Lambda(\Lambda, \lambda) = f_\Lambda^{\mathrm{ac}}(\Lambda, \lambda) - \lim_{z \to a_\Lambda(\lambda) + i0} \int_{T^1} \frac{T_\Lambda(\mu, \lambda, z)}{(a_\Lambda(\mu) - z)} f_\Lambda^{\mathrm{ac}}(\Lambda, \mu)\, d\mu \qquad \text{(E.1)}$$

Here the limit $z \to a_\Lambda(\lambda) + i0$ should be read as the limit when z tends to the point $a_\Lambda(\lambda) \in \gamma_\Lambda$ "on top," and the function $T_\Lambda(\mu, \lambda, z), \mu, \lambda \in T^1$, is an analytic extension of the function (D.2) to the complex manifold $W_{2\beta}$ with respect to the variable Λ. The existence of this extension follows from the representation (D.2), the formulas (D.3) and (D.4), and the fact that for every $\Lambda = \operatorname{Re} \Lambda + i2s$, $-G_1 < s < G_1$,

$$\Delta_\Lambda(z + i0) \neq 0$$

when $z \in \{a_\Lambda(\lambda), \lambda \in T^1\}$. In this case the function $T_\Lambda(\mu, \lambda, z), \mu, \lambda \in T^1$, is analytic in Λ and $z \notin \gamma_\Lambda$, and it has a limit when $z \to a_\Lambda(\lambda) + i0$ [except maybe the critical values $z_1(\Lambda)$ and $z_2(\Lambda)$ of the function $a_\Lambda(\lambda)$].

Analogous to the proof of the Lemma 4.2, we obtain that the above function $g_\Lambda(\Delta, \lambda)$ is analytic in Λ for every fixed $\lambda \notin \lambda_{cr}(\Lambda), \lambda'_{cr}(\Lambda)$. Thus for every fixed Λ we have to study the behavior of the function $g_\Lambda(\Lambda, \lambda)$ in the small neighborhoods O_1 and O_2 of the points $\lambda_{cr}(\Lambda)$ and $\lambda'_{cr}(\Lambda)$, respectively. In the local coordinates (Λ, ζ) of (4.23) we have two functions

$$g_\Lambda^{(i)}(\Lambda, \zeta) = g_\Lambda(\Lambda, \lambda)|_{O_i}, \qquad i = 1, 2$$

defined in the neighborhoods O_1 and O_2 respectively. The investigation of the functions $g_\Lambda^{(i)}(\Lambda, \zeta), i = 1, 2$, is made along the same lines, so we consider only the function $g_\Lambda^{(i)}(\Lambda, \zeta)$ defined in the neighborhood O_1 of the critical point $\lambda_{cr}(\Lambda)$.

Similar to the proof of Lemma 4.2, the function $\Delta_A(z)$ has the following representation in $O(z_1(A))$, where $z_1(A) = a_A(\lambda_{\mathrm{cr}}(A))$:

$$\Delta_A(z) = \frac{c_A^{(1)}(z)}{((z_1 - z)/a''_A(0))^{1/2}} + c_A^{(2)}(z) \tag{E.2}$$

Here $c_A^{(i)}(z), i = 1, 2$, are functions analytic in A, and the expression $((z_1 - z)/a''_A(0))^{1/2}$ means the branch of the function $w^{1/2}$ that has positive values on the ray $z = z_1 - a''_A(0) \cdot t, 0 < t < \infty$. In addition,

$$c_A^{(1)}(z_1(A)) \neq 0$$

An analogous representation is valid for the integral

$$\int_{T^1} \frac{D_A(\mu, \zeta, z)}{a_A(\mu) - z} f_A^{\mathrm{ac}}(A, \mu)\, d\mu = \frac{b_A^{(1)}(\zeta, z)}{((z_1 - z)/a''_A(0))^{1/2}} + b_A^{(2)}(\zeta, z) \tag{E.3}$$

where $z \in O(z_1(A))$. Here the functions $b_A^{(i)}(\zeta, z)$, $i = 1, 2$, are analytic in A, ζ, and $z \in O(z_1(A))$, and

$$b_A^{(1)}(0, z) = c_A^{(1)}(z_1(A)) \cdot f_A^{\mathrm{ac}}(A, 0) \tag{E.4}$$

Now from (E.1)–(E.4) we obtain the representation (4.24) with the condition (4.25). The lemma is proved.

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